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# Left-flat spaces with non-diverging null geodesics†

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**Abstract.** One of the two classes of non-diverging algebraically special left-flat spaces found by Fette, Janis and Newman is generalised by allowing the (self-dual part of the) Weyl tensor to become algebraically general. Penrose's conformal approach is used, thereby allowing the desired asymptotic behaviour of the solutions to be prescribed from the outset. The 'solutions' are, however, still subject to some 'reduced equations'. For a specific choice of initial data at conformal null infinity these reduced equations simplify to Plebański's second heavenly equation.

## 1. Introduction

Several years ago Fette *et al* (1977) obtained all non-diverging algebraically special left-flat spaces by means of the complexified spin-coefficient formalism. They found that the solutions split naturally into two classes, one for which the spin coefficient $\tilde{\tau}$ was zero and the other for which it was not. In this paper we extend their first class ($\tilde{\tau}=0$) by dropping the condition of algebraic speciality. However, the price we pay is that we are left with some 'reduced equations' which are, in general, difficult to solve.

The approach we take is that of Penrose's (1968) conformal method which has been used successfully several times before (see Ludwig 1976, 1978, 1980a, b) in the derivation of solutions to the Einstein equations in the real case. In particular, in one of these papers (Ludwig 1980b), the method was employed to find a further extension of Kundt's (1961) generalisation of plane gravitational waves. One of the advantages of the conformal method is that the desired asymptotic behaviour of the solutions to be found may be prescribed at the very outset. Other advantages have been discussed previously (see, e.g., Ludwig 1980b). To find solutions we transform to a conformally related space $M$ (by rescaling the metric), solve the Newman–Penrose equations there, using certain initial data defined at conformal null infinity, and then transform the result back to the original space $M$.

This conformal technique works for complex spaces as well as it does for real ones. The Newman–Penrose formalism can also be used in the study of complex space–times with only minor changes. All quantities formerly real become complex and all pairs of variables that were formerly complex conjugates now become independent (see, e.g., Fette *et al* 1977). Notationally, the bar denoting complex conjugation is replaced by a tilde.

To solve the complex Einstein vacuum equations in general is, of course, as difficult as it is in the real case. However, if we look for left-flat solutions the problem becomes

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tractable. A left-flat space, by definition, has a vanishing Ricci tensor and a self-dual Weyl tensor. The fact that the 'untilded' tetrad components $\Psi_i$ of the Weyl tensor vanish simplifies the equations enormously and allows an explicit integration of most of the equations. In this paper we obtain the metric coefficients, the spin coefficients and the Weyl tensor components for a large class of left-flat spaces in a frame based on non-expanding null geodesics. Our 'solution' is still subject to some 'reduced equations' which, for a subclass, simplify to Plebański's (1975) second heavenly equation. In the algebraically special case our frame may be so chosen that the reduced equations can be fully integrated. The resultant solution is the $\tilde{\tau}=0$ class of Fette *et al* (1977).

## 2. The choice of frame

In this section we shall set up a frame consisting of a coordinate system, one coordinate being the conformal factor, and a null tetrad. This frame will not be unique. Instead we use the frame freedom to achieve certain simplifications. But it should be emphasised that a change in frame usually involves a change in the null geodesics on which our coordinate system will be based. Therefore the transformation formulae for the coordinates will appear quite complicated except at infinity.

In Minkowski space null geodesics lying in parallel null hyperplanes all end up on the same generator $N$ of conformal future null infinity $\mathscr{I}^+$. All geodesics of a particular null hyperplane reach the same point $S$ of $N$. Actually, these null hyperplanes are null cones whose vertices lie on $N$ and for which $N$ is, in fact, one of the generators.

With this in mind we repeat the argument of a previous paper (Ludwig 1980b) and consider a space, real for the time being, which has as part of its conformal boundary a line $N$ on which the conformal factor $\Omega$ vanishes and on which $\hat{\nabla}_a\Omega\neq 0$. At a point $S$ on $N$ consider the null 'cone' generated by all the null geodesics arriving at $S$ from the interior of the rescaled space $\hat{M}$. The tangent vector $\hat{k}^a$ is defined for each such null geodesic up to a proportionality factor which depends on the geodesic. Corresponding to each $\hat{k}^a$ at $S$ choose another null vector $\hat{n}^a$ satisfying $\hat{n}_a\hat{k}^a=1$ and

$$\hat{\nabla}_a\Omega|_{\Omega=0}=K^0\hat{k}_a-\hat{n}_a \tag{2.1}$$

for some function $K^0$ also depending on the geodesic. It turns out that for vanishing Ricci tensor $K^0$ also vanishes. Hence $\hat{n}_a|_N=-\hat{\nabla}_a\Omega|_{\Omega=0}$, and $N$ is null.

For each null geodesic arriving at the point $S$ from the interior we define a tetrad $\{\hat{k}^a, \hat{m}^a, \hat{\tilde{m}}^a, \hat{n}^a\}$ at $S$ by choosing $\hat{k}^a$ and $\hat{n}^a$ as just described and $\hat{m}^a$ (and hence $\hat{\tilde{m}}^a$) arbitrarily. Propagating these tetrads parallelly into the interior of $M$ along their respective geodesics defines at each interior point (at least near $N$) precisely one tetrad. As a result of the parallel propagation of the tetrad the spin coefficients $\kappa$, $\varepsilon$, $\pi$ vanish identically.

Since each tangent vector $\hat{k}^a$ is hypersurface orthogonal $\hat{k}_a$ and $\hat{\nabla}_a u$ must be proportional for some function $u$ labelling the hypersurfaces. If this proportionality constant is set equal to one by using up some of the frame freedom we have

$$\hat{k}_a=\hat{\nabla}_a u \tag{2.2}$$

and

$$\hat{\rho}=\hat{\tilde{\rho}} \qquad \hat{\tau}=\hat{\tilde{\alpha}}+\hat{\beta}. \tag{2.3}$$

On the two-surface of intersection of an $\Omega = \text{constant} \neq 0$ hypersurface with a $u =$ constant hypersurface we select two coordinates $x$ and $y$ (or $\zeta$ and $\bar{\zeta}$, where $\zeta = -x + \mathrm{i}y$) and propagate these along the geodesics. We demand that the choice be such that $\Omega\hat{\delta}\bar{\zeta}|_{\Omega=0} = 0$, $\Omega\hat{\bar{\delta}}\zeta|_{\Omega=0} = 0$. Choosing the conformal factor as one of the coordinates completes the definition of our coordinate system $(u, \Omega, \zeta, \bar{\zeta})$.

From equations (2.1)–(2.3) and the definition of the coordinates $\zeta$ and $\bar{\zeta}$ it follows that

$$\hat{D}u = \hat{\delta}u = \hat{\bar{\delta}}u = \hat{D}\zeta = \hat{D}\bar{\zeta} = 0 \qquad \hat{\Delta}u = 1$$

and, on $N$,

$$\hat{D}\Omega = -1 \qquad \hat{\delta}\Omega = \hat{\bar{\delta}}\Omega = 0 \qquad \hat{\Delta}\Omega = K^0.$$

Hence

$$\hat{D} = \hat{f}\frac{\partial}{\partial\Omega} \qquad \hat{\Delta} = \frac{\partial}{\partial u} + \hat{U}\frac{\partial}{\partial\Omega} + \hat{X}^2\frac{\partial}{\partial\zeta} + \hat{X}^3\frac{\partial}{\partial\bar{\zeta}}$$

$$\hat{\delta} = \hat{\omega}\frac{\partial}{\partial\Omega} + \hat{\xi}^2\frac{\partial}{\partial\zeta} + \hat{\xi}^3\frac{\partial}{\partial\bar{\zeta}} \qquad \hat{\bar{\delta}} = \hat{\bar{\omega}}\frac{\partial}{\partial\Omega} + \hat{\bar{\xi}}^2\frac{\partial}{\partial\zeta} + \hat{\bar{\xi}}^3\frac{\partial}{\partial\bar{\zeta}}$$

with $\hat{f} \to -1$, $\hat{\omega} \to 0$, $\hat{\bar{\omega}} \to 0$, $\hat{U} \to K^0$ as $\Omega \to 0$. This defines the metric variables $\hat{f}$, $\hat{\omega}$, $\hat{\bar{\omega}}$, $\hat{\xi}^2$, $\hat{\xi}^3$, $\hat{\bar{\xi}}^2$, $\hat{\bar{\xi}}^3$, $\hat{U}$, $\hat{X}^2$, $\hat{X}^3$. For the real case discussed at present, the relations $\hat{X}^2 = \hat{\bar{X}}^3$, $\hat{\bar{\xi}}^2 = \hat{\bar{\xi}}^3$ and $\hat{\bar{\xi}} = \hat{\bar{\xi}}^2$ hold, but they will disappear upon complexification.

Since we are looking for divergence-free spaces we assume that the uncareted spin coefficient $\rho$ vanishes or, equivalently, that $\hat{\rho} = -\hat{f}\Omega^{-1}$. By means of some of the remaining frame freedom we can set $\hat{f} = -1$. Therefore, $\hat{D} = -\partial/\partial\Omega$ and $\hat{\rho} = \Omega^{-1}$.

Finally, defining

$$P(u, \zeta, \bar{\zeta}) = \hat{\bar{\xi}}^2\Omega|_{\Omega=0}$$

and using up some more frame freedom we make $P$ real. Near $N$,

$$\hat{\sigma} = P\Omega^{-1}\partial/\partial\zeta \qquad \hat{\bar{\sigma}} = P\Omega^{-1}\partial/\partial\bar{\zeta}.$$

Note further that at the tip of each cone we must have $\hat{\sigma} = \hat{\bar{\sigma}} = 0$.

The remaining freedom in the choice of frame is as follows.

(i) We can relabel the cones by $u' = \gamma(u)$ provided this is accompanied by a rescaling of the tetrad followed by a conformal change with parameters $a$ and $\theta$ satisfying $a^2 = \theta = \dot{\gamma}$. (For the transformation formulae see, e.g., Ludwig 1976.)

(ii) We can make a coordinate change

$$\zeta' = \zeta'(\zeta, u) \qquad \text{(and hence } \bar{\zeta}' = \bar{\zeta}'(\bar{\zeta}, u))$$

provided we also change $P$ and make a spatial rotation with parameter $\phi(u, \zeta, \bar{\zeta})$ satisfying

$$P' = P\,\mathrm{e}^{2\mathrm{i}\varphi}\partial\zeta'/\partial\zeta \qquad \text{(and } P' = P\,\mathrm{e}^{-2\mathrm{i}\varphi}\partial\bar{\zeta}'/\partial\bar{\zeta}).$$

(iii) We can make a conformal change $\Omega' = \theta\Omega$ with

$$\theta = (1 + R\Omega)^{-1} \qquad \text{where } \partial R/\partial\Omega = 0,$$

followed by a null rotation about $\hat{k}^a$ with parameter $c$ subject to

$$\partial c/\partial\Omega = \hat{\bar{\delta}}\theta \qquad \text{(and hence } \partial\bar{c}/\partial\Omega = \hat{\delta}\theta)$$

with $c \to 0$, $\bar{c} \to 0$ as $\Omega \to 0$.

The metric equations can now be worked out as usual by substituting the coordinates in the commutators of the Newman–Penrose (1962) formalism. They are

$$
\begin{aligned}
&\hat{D}\hat{U} = \hat{\gamma} + \hat{\bar{\gamma}} + \hat{\tau}\hat{\bar{\omega}} + \hat{\bar{\tau}}\hat{\omega} \\
&\hat{D}\hat{\omega} = \hat{\tau} + \hat{\sigma}\hat{\bar{\omega}} + \Omega^{-1}\hat{\omega} \\
&\hat{D}\hat{X}^{i} = \hat{\tau}\hat{\bar{\xi}}^{i} + \hat{\bar{\tau}}\hat{\xi}^{i} \qquad (i = 2, 3) \\
&\hat{D}\hat{\xi}^{i} = \hat{\sigma}\hat{\bar{\xi}}^{i} + \Omega^{-1}\hat{\xi}^{i} \\
&\hat{\delta}\hat{U} - \hat{\Delta}\hat{\omega} = \hat{\bar{\nu}} + \hat{\lambda}\hat{\bar{\omega}} + \hat{\omega}(\hat{\mu} - \hat{\gamma} + \hat{\bar{\gamma}}) \\
&\hat{\bar{\delta}}\hat{\omega} - \hat{\delta}\hat{\bar{\omega}} = \hat{\mu} - \hat{\bar{\mu}} + (\hat{\beta} - \hat{\bar{\alpha}})\hat{\bar{\omega}} + (\hat{\alpha} - \hat{\bar{\beta}})\hat{\omega} \\
&\hat{\delta}\hat{X}^{i} - \hat{\Delta}\hat{\xi}^{i} = \hat{\lambda}\hat{\bar{\xi}}^{i} + (\hat{\mu} - \hat{\gamma} + \hat{\bar{\gamma}})\hat{\xi}^{i} \\
&\hat{\bar{\delta}}\hat{\xi}^{i} - \hat{\delta}\hat{\bar{\xi}}^{i} = (\hat{\beta} - \hat{\bar{\alpha}})\hat{\bar{\xi}}^{i} + (\hat{\alpha} - \hat{\bar{\beta}})\hat{\xi}^{i}.
\end{aligned}
\tag{2.4}
$$

Note that although not explicitly written down, the complex conjugates of equations (2.4) must also be considered. Thus, for example,

$$\hat{D}\hat{\bar{\omega}} = \hat{\bar{\tau}} + \hat{\bar{\sigma}}\hat{\omega} + \Omega^{-1}\hat{\bar{\omega}}.$$

## 3. The rescaled left-flat space

Upon complexification the discussion of § 2 goes through essentially unaltered. As usual, the bar over a variable is replaced by a tilde to emphasise that this new variable is independent of the original one. The equations at our disposal are as follows.

(1) The Ricci identities (see Newman and Penrose 1962), e.g.

$$\hat{D}\hat{\tau} - \hat{\Delta}\hat{\kappa} = \hat{\rho}(\hat{\tau} + \hat{\tilde{\pi}}) + \hat{\sigma}(\hat{\tilde{\tau}} + \hat{\pi}) + \hat{\tau}(\hat{\varepsilon} - \hat{\tilde{\varepsilon}}) - \hat{\kappa}(3\hat{\gamma} + \hat{\tilde{\gamma}}) + \hat{\Psi}_1 + \hat{\varphi}_{01}.$$

(2) The metric equations (2.4).

(3) The transformation equations for the Ricci tensor (see Ludwig 1976), e.g.

$$\varphi_{01} = \Omega^3\hat{\varphi}_{01} + \Omega^2[\hat{D}\hat{\delta}\Omega - \hat{\tilde{\pi}}\hat{D}\Omega + (\hat{\tilde{\varepsilon}} - \hat{\varepsilon})\hat{\delta}\Omega + \hat{\kappa}\hat{\Delta}\Omega].$$

Again, it should be remembered that along with these equations we must consider their 'tilded' versions.

Since in this paper we are looking for left-flat spaces the components $\varphi_{00}$, $\varphi_{01}$, $\varphi_{10}$, etc, of the ('uncareted') Ricci tensor and the components $\Psi_i$ (or $\hat{\Psi}_i$) of the Weyl tensor vanish. (Of course, the components $\hat{\varphi}_{00}$, etc, of the 'careted' Ricci tensor and the 'tilded' components $\tilde{\Psi}_i$ (or $\hat{\tilde{\Psi}}_i$) of the Weyl tensor are not zero in general.) This leads to considerable simplification making it possible to carry out to a large extent the integration of the abovementioned equations. However, we will still be left with a handful of 'reduced equations' which must be solved before an actual solution is obtained. The results of this straightforward but tedious calculation are as follows. For notational reasons we shall leave out all carets *in this set of equations only.* All variables involved refer, however, to the space conformally related to the left-flat space we are trying to find.

*Metric coefficients*

$$f=-1 \qquad \omega=0 \qquad \tilde{\omega}=B \qquad \xi^2=\tilde{\xi}^3=-2P\Omega^{-1} \qquad \xi^3=0$$
$$\tilde{\xi}^2=-2P\Omega G' \qquad U=U^{(1)}\Omega+\Omega^2(U^{(2)}+\eth^2 T+\tilde{\lambda}^0 G) \tag{3.1}$$
$$X^2=X-2P(\eth G+\Omega^{-1}\eth H) \qquad X^3=\tilde{X}.$$

*Spin coefficients*

$$0=\kappa=\varepsilon=\pi=\sigma=\tau=\tilde{\kappa}=\tilde{\varepsilon}=\tilde{\pi} \qquad \rho=\tilde{\rho}=\Omega^{-1} \qquad \tilde{\lambda}=\tilde{\lambda}^0$$
$$\tilde{\sigma}=-(\Omega^2 G')' \qquad \tilde{\tau}=-\Omega^{-1}B-\Omega\eth G'$$
$$\tilde{\alpha}=\Omega^{-1}\tilde{\alpha}^0 \qquad \beta=-\Omega^{-1}\tilde{\alpha}^0 \qquad \alpha=\alpha^0\Omega^{-1}-B\Omega^{-1}-\tilde{\alpha}^0\Omega G'$$
$$\tilde{\beta}=-\Omega\eth G'-\alpha^0\Omega^{-1}+\tilde{\alpha}^0\Omega G'$$
$$\gamma=\gamma^0-\Omega(U^{(2)}+\eth^2 T+\tilde{\lambda}^0 G)-\tilde{\alpha}^0(\eth G+\Omega^{-1}\eth H)$$
$$\tilde{\gamma}=\tilde{\gamma}^0-\Omega(U^{(2)}+\eth^2 T+\tilde{\lambda}^0 G)+\tilde{\alpha}^0(\eth G+\Omega^{-1}\eth H)-\tilde{\lambda}^0\Omega^2 G'-\eth^2 G$$
$$\mu=\mu^0+\Omega(U^{(2)}+\eth^2 T+\tilde{\lambda}^0 G)-\Omega^{-1}\eth^2 H \tag{3.2}$$
$$\tilde{\mu}=\mu^0+\Omega(U^{(2)}+\eth^2 T+\tilde{\lambda}^0 G)-\eth^2 G$$
$$\tilde{\nu}=\tilde{\lambda}^0(\eth H-2\Omega\eth G)-\eth U^{(1)}-\Omega(\eth U^{(2)}+\eth^3 T+G\eth\tilde{\lambda}^0)$$
$$\lambda=\Omega^2 G'(\mu^0-U^{(1)}-\Omega^{-1}\eth^2 H)+\Omega^{-1}B^2-\Omega^{-1}\bar{\eth}\eth H+\lambda^0$$
$$\nu=B[\Omega^{-1}U-\mu^0+\eth^2 G+\tfrac{1}{2}\eth(X/2P)-\tfrac{1}{2}\bar{\eth}(\tilde{X}/2P)]-\Omega G'\eth U$$
$$-\Omega U\eth G'-\Omega^{-1}\bar{\eth}U+(\eth G+\Omega^{-1}\eth H)\eth B-(X/2P)\eth B-(\tilde{X}/2P)\bar{\eth}B-\dot{B}.$$

*Weyl tensor*

$$0=\Psi_0=\Psi_1=\Psi_2=\Psi_3=\Psi_4$$
$$\tilde{\Psi}_0=(\Omega^2 G)''' \qquad \tilde{\Psi}_1=\Omega^{-1}\eth(\Omega^2 G')' \qquad \tilde{\Psi}_2=\eth^2 G'+\tilde{\lambda}^0(\Omega^2 G')'$$
$$\tilde{\Psi}_3=-\Omega^{-1}\eth U^{(1)}+3\tilde{\lambda}^0\Omega\eth G'+\Omega G'\eth\tilde{\lambda}^0+\Omega^{-1}\eth^3 G \tag{3.3}$$
$$\tilde{\Psi}_4=-\Omega^{-1}\eth\tilde{\nu}-\dot{\tilde{\lambda}}^0-(X/2P)\eth\tilde{\lambda}^0-(\tilde{X}/2P)\bar{\eth}\tilde{\lambda}^0+(\eth G+\Omega^{-1}\eth H)\eth\tilde{\lambda}^0$$
$$+\tilde{\lambda}^0[U^{(1)}-2\mu^0-\eth(X/2P)+\bar{\eth}(\tilde{X}/2P)+4\eth^2 G+\Omega^{-1}\eth^2 H+3\tilde{\lambda}^0\Omega^2 G'].$$

*Ricci tensor*

$$\varphi_{00}=\varphi_{01}=0 \qquad \varphi_{10}=\eth G' \qquad \varphi_{02}=-\tilde{\lambda}^0\Omega^{-1}$$
$$\varphi_{11}=-\tfrac{1}{2}\mu^0\Omega^{-1}+\Omega^{-1}\eth^2 G+\tfrac{1}{2}\tilde{\lambda}^0\Omega G'$$
$$\varphi_{12}=\eth^3 T+\Omega^{-1}\eth U^{(1)}+\eth U^{(2)}+\tilde{\lambda}^0(2\eth G-\Omega^{-1}\eth H)+G\eth\tilde{\lambda}^0$$
$$\varphi_{20}=-\tilde{\sigma}\Omega^{-1}U-\Omega^{-1}\lambda^0+\Omega^{-1}\bar{\eth}\eth G-2B\eth G'+\Omega G'(U^{(1)}-\mu^0+\eth^2 G) \tag{3.4}$$
$$\varphi_{21}=\Omega^{-1}B(\mu^0-\tilde{\lambda}^0\Omega^2 G'-2\eth^2 G)+U\eth G'+G'\eth U+\Omega^{-2}\bar{\eth}U$$
$$\varphi_{22}=-\Omega^{-1}\dot{U}+\Omega^{-1}B\tilde{\nu}-\Omega^{-1}(X/2P)\eth U-\Omega^{-1}(\tilde{X}/2P)\bar{\eth}U+\Omega^{-1}(\eth G+\Omega^{-1}\eth H)\eth U$$
$$\Lambda=-U^{(2)}-\tfrac{1}{2}\mu^0\Omega^{-1}-\eth^2 T-\tfrac{1}{2}\tilde{\lambda}^0\Omega^{-1}(\Omega^2 G)'$$

where

$$\alpha^0=-\partial P/\partial\tilde{\zeta} \qquad \tilde{\alpha}^0=-\partial P/\partial\zeta \qquad \lambda^0=\partial X/\partial\tilde{\zeta}=\bar{\eth}(X/2P)$$

$$\tilde{\lambda}^0 = \partial\tilde{X}/\partial\zeta = \eth(\tilde{X}/2P) \qquad \gamma^0 + \tilde{\gamma}^0 = -U^{(1)}$$

$$\tilde{\gamma}^0 - \gamma^0 = \frac{1}{2}\frac{\partial X}{\partial \zeta} - \frac{1}{2}\frac{\partial \tilde{X}}{\partial \tilde{\zeta}} = \tfrac{1}{2}\eth\left(\frac{X}{2P}\right) - \tfrac{1}{2}\tilde{\eth}\left(\frac{\tilde{X}}{2P}\right) - \tilde{\alpha}^0\frac{X}{P} + \alpha^0\frac{\tilde{X}}{P}$$

$$B = \Omega\eth G - \eth H$$

$$H' = G \qquad T' = \Omega^{-2}G.$$

The variables $\mu^0$, $U^{(1)}$, $U^{(2)}$, $P$, $X$, $\tilde{X}$, all functions of $u$, $\zeta$, $\tilde{\zeta}$, and the function $G(u, \Omega, \zeta, \tilde{\zeta})$ are subject to the 'reduced equations'

$$0 = \partial^2 \ln P/\partial\zeta\partial\tilde{\zeta} \qquad 0 = \tilde{\eth}U^{(1)} \qquad 0 = \eth\lambda^0 - \tilde{\eth}\mu^0$$

$$0 = \eth\mu^0 - 2\eth U^{(1)} - \tilde{\eth}\tilde{\lambda}^0$$

$$\dot{P}P^{-1} = U^{(1)} - \mu^0 + \tfrac{1}{2}\eth(X/2P) + \tfrac{1}{2}\tilde{\eth}(\tilde{X}/2P) \tag{3.5}$$

$$\dot{U}^{(1)} - \dot{\mu}^0 = \lambda^0\tilde{\lambda}^0 - \tilde{\eth}\eth U^{(2)} + \mu^0(\mu^0 - U^{(1)}) + (\tilde{X}/2P)\tilde{\eth}\mu^0 + (X/2P)(\eth\mu^0 - \eth U^{(1)})$$

$$\dot{\lambda}^0 = \lambda^0[U^{(1)} - 2\mu^0 + \eth(X/2P) - \tilde{\eth}(\tilde{X}/2P)] + \tilde{\eth}^2U^{(2)} - (X/2P)\eth\lambda^0 - (\tilde{X}/2P)\tilde{\eth}\lambda^0$$

$$\dot{G} = -UG' + (\eth G)^2 - \tilde{\eth}\eth T - (X/2P)\eth G - (\tilde{X}/2P)\tilde{\eth}G + G[\eth(X/2P) - \tilde{\eth}(\tilde{X}/2P) - \mu^0].$$

The dot and prime refer to differentiation with respect to the coordinates $u$ and $\Omega$ respectively. The differential operators $\eth$ and $\tilde{\eth}$ are defined by

$$\eth\eta = 2P(\partial\eta/\partial\zeta) - 2s\tilde{\alpha}^0\eta$$

$$\tilde{\eth}\eta = 2P(\partial\eta/\partial\tilde{\zeta}) + 2s\alpha^0\eta$$

where $s$ is the spin weight of $\eta$.

## 4. Left-flat space

Converting back to the actual left-flat space we are looking for, we introduce coordinates $(u, r, \zeta, \tilde{\zeta})$ there as follows. The coordinates $u$, $\zeta$, $\tilde{\zeta}$ are the same as in the rescaled space. Since, as usual, we want $D = \partial/\partial r$ we determine the coordinate $r$ from the equation

$$\partial\Omega/\partial r = D\Omega = \Omega^2\hat{D}\Omega = -\Omega^2.$$

Therefore $r = \Omega^{-1}$ if the origin is suitably chosen. it should be remembered (see the appendix to Ludwig 1976) that in order to keep the tetrad parallelly propagated the conformal rescaling has to be followed by a null rotation of the tetrad about $k^a$. It turns out that the parameters of this null rotation, $c$ and $\tilde{c}$, must be $-\Omega^{-1}\eth H$ and zero, respectively. The transformation formulae for the various variables of the Newman–Penrose formalism are standard (see Ludwig 1976). After some calculation we obtain the following final results.

*Metric coefficients*

$$\omega = 0 \qquad \tilde{\omega} = -\eth G$$

$$\xi^2 = \tilde{\xi}^3 = -2P \qquad \xi^3 = 0 \qquad \tilde{\xi}^2 = PG_r$$

$$U = -r\hat{U}^{(1)} - \hat{U}^{(2)} - \eth^2T - \hat{\tilde{\lambda}}^0G \tag{4.1}$$

$$X^2 = \hat{X} - 2P\eth G \qquad X^3 = \hat{\tilde{X}}.$$

*Spin coefficients*

$$
\begin{aligned}
&0=\kappa=\varepsilon=\pi=\rho=\sigma=\tau=\tilde\kappa=\tilde\varepsilon=\tilde\pi=\tilde\rho\\
&\alpha=\hat\alpha^0+\hat{\hat\alpha}{}^0G_r \qquad \beta=-\hat{\hat\alpha}{}^0 \qquad \gamma=\hat\gamma^0+\hat U^{(1)}-\hat{\hat\alpha}{}^0\eth G\\
&\mu=\hat\mu^0-\hat U^{(1)} \qquad \lambda=\hat\lambda^0-(\hat\mu^0-\hat U^{(1)})G_r \qquad \nu=-\bar\eth\hat U^{(2)}+(\hat\mu^0-\hat U^{(1)})\eth G\\
&\tilde\sigma=-G_{rr} \qquad \tilde\tau=\eth G_r \qquad \tilde\alpha=\hat{\hat\alpha}{}^0 \qquad \tilde\beta=-\hat\alpha^0-\hat\alpha^0G_r+\eth G_r \qquad (4.2)\\
&\tilde\gamma=\hat{\hat\gamma}{}^0+\hat U^{(1)}-\eth^2G+\hat{\hat\lambda}{}^0G_r+\hat{\hat\alpha}{}^0\eth G\\
&\tilde\mu=\hat\mu^0-\hat U^{(1)}-\eth^2G \qquad \tilde\lambda=\hat{\hat\lambda}{}^0\\
&\tilde\nu=-r\eth\hat U^{(1)}-\eth\hat U^{(2)}-\eth^3T-G\eth\hat{\hat\lambda}{}^0-2\hat{\hat\lambda}{}^0\eth G.
\end{aligned}
$$

*Weyl tensor*

$$
\begin{aligned}
&\tilde\Psi_0=-G_{rrr} \qquad \tilde\Psi_1=\eth G_{rr} \qquad \tilde\Psi_2=-\eth^2G_r+\hat{\hat\lambda}{}^0G_{rr}\\
&\tilde\Psi_3=-\eth\hat U^{(1)}+\eth^3G-3\hat{\hat\lambda}{}^0\eth G_r-G_r\eth\lambda^0\\
&\tilde\Psi_4=-\eth^2U+\hat{\hat\lambda}{}^0[5\eth^2G-3\hat{\hat\lambda}{}^0G_r+\hat U^{(1)}-2\hat\mu^0-\eth(\hat X/2P)+\bar\eth(\hat{\hat X}/2P)]\\
&\qquad +2\eth\hat{\hat\lambda}{}^0(\eth G)-\dot{\hat\lambda}{}^0-(\hat X/2P)\eth\hat{\hat\lambda}{}^0-(\hat{\hat X}/2P)\bar\eth\hat{\hat\lambda}{}^0
\end{aligned}
\qquad (4.3)
$$

where

$$
\begin{aligned}
&\hat\alpha^0=-\partial P/\partial\tilde\zeta \qquad \hat{\hat\alpha}{}^0=-\partial P/\partial\zeta \qquad \hat\lambda^0=\partial\hat X/\partial\tilde\zeta \qquad \hat{\hat\lambda}{}^0=\partial\hat{\hat X}/\partial\zeta\\
&\hat\gamma^0+\hat{\hat\gamma}{}^0=-\hat U^{(1)} \qquad \hat{\hat\gamma}{}^0-\hat\gamma^0=\frac{1}{2}\frac{\partial\hat X}{\partial\zeta}-\frac{1}{2}\frac{\partial\hat{\hat X}}{\partial\tilde\zeta} \qquad T_r=-G
\end{aligned}
\qquad (4.4)
$$

and the subscript $r$ denotes differentiation with respect to $r$.

The functions $G$, $\hat\mu^0$, $\hat U^{(1)}$, $\hat U^{(2)}$, $P$, $\hat X$ and $\hat{\hat X}$ are subject to the 'reduced equations' given by equations (3.5). The last one of these, in terms of $r$ and an 'uncareted' $U$, becomes

$$\dot G=-UG_r+(\eth G)^2-\bar\eth\eth T-(\hat X/2P)\eth G-(\hat{\hat X}/2P)\bar\eth G+G[\eth(\hat X/2P)-\bar\eth(\hat{\hat X}/2P)-\hat\mu^0].$$

These final results can be (and have been) checked by direct substitution into the 'uncareted' equations. In the case of vanishing $G$, i.e. the algebraically special case, they can also be deduced (with but a slight change in notation) from Ludwig (1980b) by setting the $\Psi_i$ equal to zero.

From the metric variables we can calculate (see Newman and Penrose 1962) the contravariant metric tensor. It is given by

$$(g^{\alpha\beta})=\begin{pmatrix}0&1&0&0\\1&2U&\hat X-4P\eth G&\hat{\hat X}\\0&\hat X-4P\eth G&8P^2G_r&-4P^2\\0&\hat{\hat X}&-4P^2&0\end{pmatrix}.$$

By matrix inversion we find

$$(g_{\alpha\beta})=\begin{pmatrix}g_{11}&1&\hat{\hat X}/4P^2&g_{14}\\1&0&0&0\\\hat{\hat X}/4P^2&0&0&-(4P^2)^{-1}\\g_{14}&0&-(4P^2)^{-1}&-(2P^2)^{-1}G_r\end{pmatrix}$$

where

$$g_{11} = -2U + (2P^2)^{-1}\hat{\hat{X}}(-G_r\hat{X} - \hat{X} + 4P\bar{\eth}G)$$
$$g_{14} = -(4P^2)^{-1}(-2G_r\hat{\hat{X}} - \hat{X} + 4P\bar{\eth}G).$$

Therefore the metric is given by

$$ds^2 = 2\,du\,dr + g_{11}\,du^2 + (2P^2)^{-1}\hat{\hat{X}}\,d\zeta\,du + 2g_{14}\,d\tilde{\zeta}\,du$$
$$-(2P^2)^{-1}\,d\zeta\,d\tilde{\zeta} - (2P^2)^{-1}G_r\,d\tilde{\zeta}^2. \tag{4.5}$$

## 5. Discussion

There is still considerable freedom in the choice of frame. We can use part of it to make $P$ equal to a constant. If we choose this constant to be $\frac{1}{2}$ then $\eth = \partial/\partial\zeta$ and $\tilde{\eth} = \partial/\partial\tilde{\zeta}$.

Let us consider the algebraically special case; then $G$ vanishes. The 'reduced equations' (3.5) are now fairly easy to integrate. We can use the remaining frame freedom to eliminate a number of the 'constants' of integration. We can also arrange the frame so that $\hat{\mu}^0 = \hat{U}^{(1)}$. The details are straightforward. We find that

$$X^2 = \hat{X}(u, \zeta) \qquad X^3 = -\tilde{\zeta}\eth\hat{X} \qquad \hat{\mu}^0 = \hat{U}^{(1)} = \eth\hat{X} \qquad \hat{U}^{(2)} = \hat{U}^{(2)}(u, \zeta).$$

The variables $\hat{X}$ and $\hat{U}^{(2)}$ remain arbitrary functions of the two coordinates $u$ and $\zeta$. Substitution of these values into equations (4.1)–(4.4) reduces the latter to the $\tilde{\tau} = 0$ class of non-diverging left-flat spaces found by Fette *et al* (1977). (Note, however, that in the general case, with $G \neq 0$, the spin coefficient $\tilde{\tau}$ does not vanish.) The remaining frame freedom is now as given in § 2, with

(i) $\gamma$ restricted to be a linear function of $u$;

(ii) $\varphi$, $\zeta'$ and $\tilde{\zeta}'$ restricted by

$$1 = e^{2i\varphi}\partial\zeta'/\partial\zeta = e^{-2i\varphi}\partial\tilde{\zeta}'/\partial\tilde{\zeta};$$

(iii) $R$ restricted by

$$R = c_1\tilde{\zeta} + r(u)$$

where $c_1$ is a constant and $r(u)$ is an arbitrary function of $u$.

Finally, if in the general solution, equations (4.1)–(4.4), we take

$$P = \hat{U}^{(2)} = 1 \qquad \hat{U}^{(1)} = \hat{X} = \hat{\hat{X}} = 0 \qquad T = -\theta$$

then the metric, given by equation (4.5), becomes

$$ds^2 = 2\,du\,dr + (2 - 8\theta_{\zeta\zeta})\,du^2 - 4\theta_{r\zeta}\,du\,d\tilde{\zeta} - \tfrac{1}{2}\theta_{rr}\,d\tilde{\zeta}^2 - \tfrac{1}{2}\,d\zeta\,d\tilde{\zeta}$$

and the only surviving 'reduced equation' is Plebański's (1975) 'second heavenly equation'

$$4\theta_{rr}\theta_{\zeta\zeta} - 4(\theta_{r\zeta})^2 + \theta_{ru} - \theta_{rr} - 4\theta_{\zeta\tilde{\zeta}} = 0.$$

If we specialise further and take $\theta = \lambda/2\delta$, where $\delta = ur + u^2 - \frac{1}{4}\zeta\tilde{\zeta}$ we obtain Sparling and Tod's (1981) $\mathscr{H}$ space, albeit in a frame different from the usual one.

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## References

Fette C W, Janis A I and Newman E T 1977 *Gen. Rel. Grav.* **8** 29–43
Kundt W 1961 *Z. Phys.* **163** 77–86
Ludwig G 1976 *Gen. Rel. Grav.* **7** 293–311
——1978 *Gen. Rel. Grav.* **9** 1009–19
——1980a *J. Math. Phys.* **21** 154–60
——1980b *J. Math. Phys.* **21** 2543–8
Newman E T and Penrose R 1962 *J. Math. Phys.* **3** 566–78
Penrose R 1968 *Battelle Rencontres* ed C De Witt (New York: Benjamin) pp 171–86
Plebański J F 1975 *J. Math. Phys.* **16** 2395–402
Sparling G A J and Tod K P 1981 *J. Math. Phys.* **22** 331–2